Quivers, algebras and representation type

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Acknowledgement. Much of the material herein has been taken from lectures given by de la Peña at the *Advanced school and conference on representation theory and related topics* in 2006. The original notes can (at the time of writing) be found at `http://www.math.jussieu.fr/~keller/ictp2006/lecturenotes/`.

1 Conventions

Throughout, k will denote an algebraically closed field.

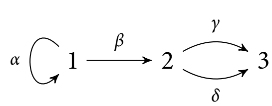
There will probably be other conventions alongside definitions, but I guarantee nothing.

2 Quivers

Definition (Quiver)**.** A **quiver** $Q = (Q_0, Q_1)$ is a directed graph on vertices Q_0 with edges Q_1.

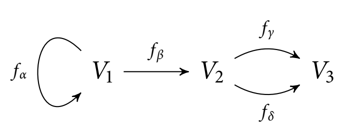
Though it is not strictly part of the definition, we usually assume that a quiver is connected.

Example.

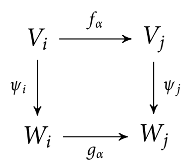


Definition (Representation of a quiver)**.** A **representation** of a quiver Q, consists of a vector space, V_i for each vertex $i \in Q_0$ and a linear map $f_\alpha \colon V_i \to V_j$ for each edge $\alpha \colon i \to j$ in Q_1. We denote the representation by (V, f).

Example.

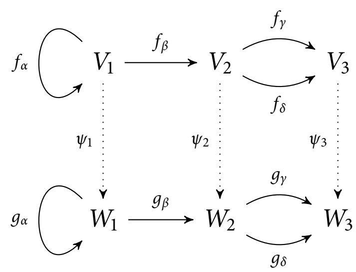


Definition (Morphism of representations)**.** A morphism $\psi \colon (V, f) \to (W, g)$ is consists of a map $\psi_i \colon V_i \to W_i$ for each $i \in Q_0$, such that the squares



commute for all $\alpha \colon i \to j$ in Q_1.

Example.



Compare all of the above to definitions in category theory. It is clear that a quiver generates a category, by composing edges into paths. In this case our representation is simply a functor from the category of Q to the category of vector spaces. Our morphisms are then natural transformations between these functors.

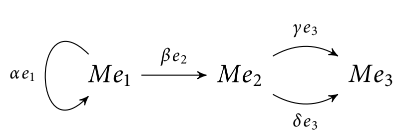
There is a correspondence, for example in group theory, between the representations of a group and modules for the group algebra. The same is true in the case of quiver representations. We require an algebra in order to define our modules.

A module is a generalisation of a vector space, for which the base space need only be a ring rather than a field.

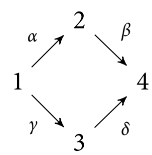
Definition (Path algebra)**.** The **path algebra** kQ defined to be the algebra whose underlying vector space is generated by the paths in Q including a zero length path starting at each vertex. The multiplication of two paths is given by concatenation if this makes sense (that is the first path ends where the second begins) and is zero otherwise. This multiplication is then extended linearly.

An algebra over a field is a ring that admits a distributive scalar multiplication from the base field; it is a ring and a vector space.

It is then true that a representation of the quiver is the same as a module for the path algebra. For example, let M be a module for the algebra kQ we generate the representation:



We can also put relations on top of our quivers for example



$\alpha\beta = \gamma\delta$

Given such a **bound quiver** or **quiver with relations** we vary our definitions in the obvious way. That is, for a representation (V, f) we must have $f_\beta f_\alpha = f_\delta f_\gamma$.

The path algebra would be defined as the quotient of the unbounded path algebra by the ideal generated by the relations:

$$\frac{kQ}{\langle \alpha\beta - \gamma\delta \rangle}$$

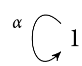
Let $J = \mathrm{rad}(kQ) = \langle Q_1 \rangle$ be the ideal generated by all the edges; that is everything except zero length paths. We consider relations I admissible if $I \leq J^2$. If we allowed a length one relation to appear then we could remove an edge from the quiver to obtain the same path algebra. Often we also require that $J^n \leq I$, which ensures that our path algebra is finite dimensional.

Theorem 2.1:

> Every finite dimensional algebra is Morita equivalent to the algebra of a finite quiver with admissible relations.

By Morita equivalence we mean that the module categories are equivalent, so that from a representation theory point of view the algebras are the same.

Example. Consider C_n, the cyclic group of order $n = p^t$, where p is the characteristic of the field k, and the quiver

$Q =$ 

$I = \langle \alpha^n \rangle$

Then $kC_n \cong kQ/I \cong k[x]/(x^n)$.

3 Representation type

3.1 Finite representation type

Consider the algebra $A = k[x]/(x^n)$.

A module M for A is fully determined by the action of x on a M, thus we can think of M simply as a square matrix with entries in k. We have that $x^n = 0$ and thus $M^n = 0$.

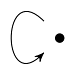
Thus we can write M in the form $J_{n_1}(0) \oplus J_{n_2}(0) \oplus \cdots \oplus J_{n_s}(0)$, where $J_n(\lambda)$ is the $n \times n$ Jordan block with eigenvalue λ and $n_i \leq n$ for all i.

If we further assume that M is indecomposable[1] then we have that $M \cong J_s(0)$ for some $s \leq n$. This shows that A has only finitely many non-isomorphic indecomposable modules. Such an algebra is said to have **finite representation type**.

[1] A module is indecomposable if it cannot be written as the direct sum of two non-zero modules.

3.2 Tame representation type

Consider the algebra $A = k[x]$, which is the algebra for the quiver



without relations.

As for the case above a module is given by an endomorphism of a vector space. The difference from the case with relations is that we no longer require this endomorphism to be nilpotent.

We can write M in the form $J_{n_1}(\lambda_1) \oplus J_{n_2}(\lambda_2) \oplus \cdots \oplus J_{n_s}(\lambda_s)$ where $(x-\lambda_1)^{n_1} \ldots (x-\lambda_s)^{n_s}$ is the characteristic polynomial for M.

Consider the $k[x]$-$k[x]$–bimodule

$$J_n(x) = \begin{bmatrix} x & 1 & \cdots & 0 \\ 0 & x & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x \end{bmatrix}$$

and let $S_\lambda = k[x]/(x-\lambda)$, a simple 1-dimensional $k[x]$-module. We have that

S_λ is simply λ in its sum of Jordan blocks form.

$$J_n(x) \underset{k[x]}{\otimes} S_\lambda = J_n(\lambda)$$

The image of the functor

$$J_n(x) \underset{k[x]}{\otimes} -\colon \operatorname{mod} k[x](1) \longrightarrow \operatorname{mod} k[x]$$

from 1-dimensional $k[x]$-modules to $k[x]$-modules contains (an isomorphic copy) of every n-dimensional indecomposable $k[x]$ module.

The way to think about what is happening here is that the n-dimensional modules are being covered by a one parameter family of modules. That is each n-dimensional module is of the form $J_n(\lambda)$ for λ in k. This characterises what it means for an algebra to have **tame representation type**.

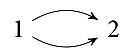
Definition (Tame representation type)**.** An algebra A, is **tame** if it is not of finite representation type and if for each $n \in \mathbb{N}$ there is a finite family of A-$k[t]$–bimodules $M_1, \ldots, M_{s(n)}$ such that

(i) M_i is finitely-generated and free as $k[t]$-module;

(ii) for almost all indecomposable A-modules, X, of dimension n we have $X \cong M_i \underset{k[t]}{\otimes} S_\lambda$ for some $1 \le i \le s(n)$ and for some $\lambda \in k$.

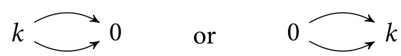
This definition means that in each dimension, the indecomposable modules are covered by a finite set of 1-parameter families of modules.

Example. Consider the quiver without relations

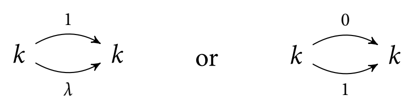


The indecomposable modules are covered by the families

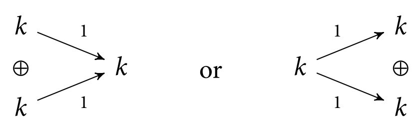
dim = 1



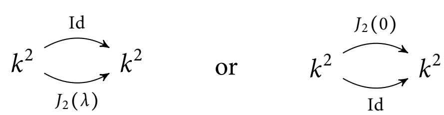
dim = 2



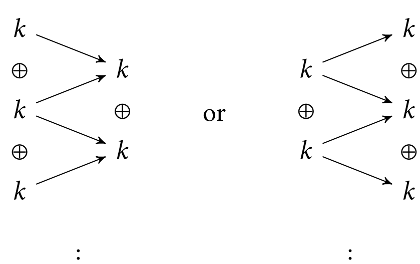
dim = 3



dim = 4



dim = 5



3.3 Wild representation type

Consider the algebra $k\langle u, v\rangle$, the group algebra for the free group on two generators. There is a problem with this algebra:

Theorem 3.1:

> For any finitely generated algebra A over k there is a fully faithful functor $\operatorname{mod} A \to \operatorname{mod} k\langle u, v\rangle$

Proof. Since A is finitely generated, say $A = \langle a_1, \dots, a_s \rangle$ and let M be the left A-module A^{s+2}. We let u and v act on the right of M via the linear maps

$$u = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & & \ddots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \ddots & & 0 \\ a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & a_s & 1 & 0 \end{bmatrix}$$

the functor $M \underset{A}{\otimes} - \colon \operatorname{mod} A \to \operatorname{mod} k\langle u, v \rangle$ is fully faithful. The proof of this is straightforward yet time consuming. □

Thus we see that the algebra $k\langle u, v \rangle$ contains the representation theory for all finitely generated k-algebras. This is the archetypical **wild algebra**.

Definition (Wild representation type)**.** An algebra A is said to have **wild representation type** if there is a $k\langle u, v \rangle$-A bimodule, M, such that

(i) M is free as a $k\langle u, v \rangle$-module;

(ii) if X is an indecomposable $k\langle u, v \rangle$-module then $M \otimes X$ is indecomposable;

(iii) if $M \otimes X \cong M \otimes Y$ then $X \cong Y$

A functor satisfying (i–iii) is said to **inset** indecomposable modules.

If further the module category of $k\langle u, v \rangle$ embeds fully faithfully into the module category of A we say that A is **strictly wild**.

Example. Consider the quiver without relations, Q:

$$1 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; 2$$

The representation

$$k\langle u, v \rangle \;\substack{\xrightarrow{\;1\;} \\ \xrightarrow[\;u\;]{} \\ \xrightarrow[\;v\;]{}}\; k\langle u, v \rangle$$

is a $k\langle u, v \rangle$-kQ–bimodule satisfying the properties of the definition.

Notice that here we abuse the correspondence between representations and modules further in that we denote a bimodule as a representation of kQ with $k\langle u, v \rangle$-modules in place of vector spaces.

[Dro80] Drozd, *Tame and wild matrix problems*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 242–258

[CB88] Crawley-Boevey, *On tame algebras and bocses*, Proc. London Math. Soc. (3) **56** (1988), no. 3, 451–483

Theorem 3.2: *(Drozd, 1977)*

> Every finite-dimensional algebra over an algebraically closed field has exactly one representation type: finite, tame or wild.

The proof of this result is way beyond the scope of this talk (and the author's abilities). See [Dro80] or [CB88] for details.

4 Further topics

Let Q be a connected quiver without oriented cycles. Such a quiver has the property that kQ is hereditary. We denote by $|Q|$ the undirected graph of Q.

Consider the matrix

$$M_Q = (m_{ij}) = \begin{cases} 2 & i = j \\ -(\#\text{ edges between } i \text{ and } j) & i \neq j \end{cases}$$

the incidence matrix for $|Q|$.

We consider the action of this matrix on $\mathbb{Z}^n$, where n is the number of vertices in Q.

Claim. $M_Q^{-1}(\mathbb{N}^n) \cap \partial(\mathbb{N}^n) = \{0\}$, where ∂ is the boudary operator.

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \partial(\mathbb{N}^n)$ be such that $M\mathbf{x} \in \mathbb{N}^n$.

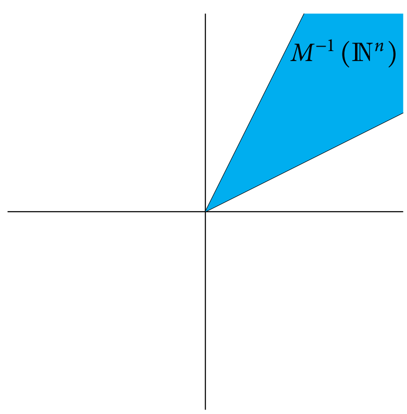
Since Q is connected there is an edge $i - j$ such that $x_i = 0$ but $x_j > 0$. We have

$$\begin{aligned} (M\mathbf{x})_i &= \sum_k m_{ik}x_k \\ &= m_{ij}x_j + \sum_{k \neq i,j} m_{ik}x_k \\ &\leq m_{ij}x_j < 0 \end{aligned}$$

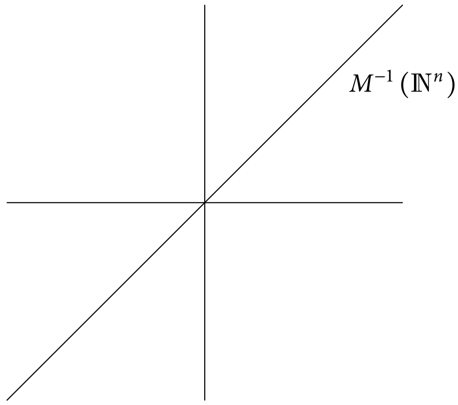
□

These gives three possibilities for the action of M.

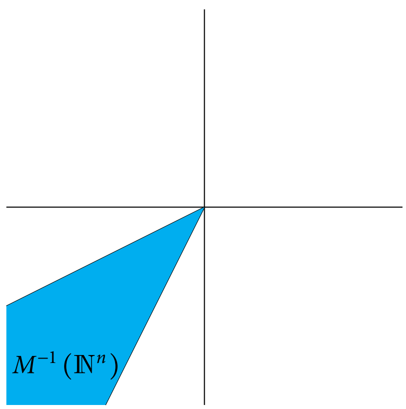
(a) $M_Q^{-1}(\mathbb{N}^n) \subseteq \mathbb{N}^n_{>0} \cup \{0\}$



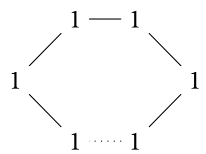
(b) $M_Q^{-1}(\mathbb{N}^n) = \mathbb{Z}\mathbf{u}$ for some $\mathbf{u} \in \mathbb{Z}^n$. In this case $M\mathbf{u} = 0$



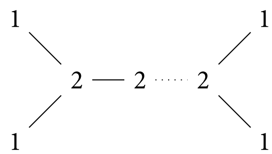
(c) $M_Q^{-1}(\mathbb{N}^n) \cap \mathbb{N}^n = \{0\}$



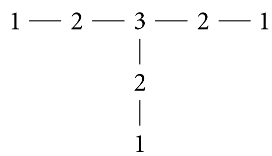
The following quivers are of type (b):

$\tilde{A}_n$ = 

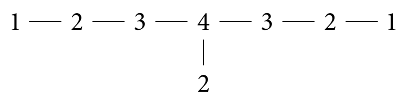
with $n + 1$ vertices

$\tilde{D}_n$ = 

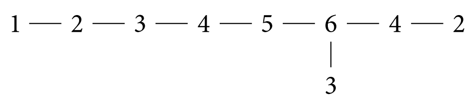
with $n + 1$ vertices

$\tilde{E}_6$ = 

with 7 vertices

$\tilde{E}_7$ = 

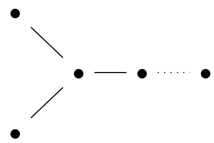
with 8 vertices

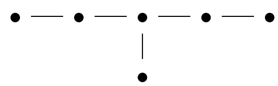
$\tilde{E}_8$ = 

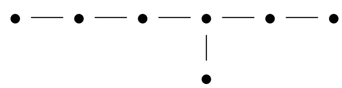
with 9 vertices

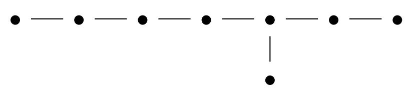
The quivers that contain non of the type (b) quivers are of type (a). These are the quivers:

A_n = with n vertices

D_n =  with n vertices

E_6 =  with 6 vertices

E_7 =  with 7 vertices

E_8 =  with 8 vertices

All other quivers are of type (c).

Theorem 4.1:

Quivers of type

(a) are of finite representation type;

(b) are of tame representation type;

(c) are of wild representation type.

This idea generalises to the case where we have relations, though still no oriented cycles. Let R be a minimal generating set for the relations such that $R \subseteq \bigcup_{i,j \in Q_0} I(i,j)$, where $I(i,j)$ denotes the ideal consisting of paths starting at i and ending at j. Let $r_{ij} = |R \cap I(i,j)|$. We define the quadratic form

$$q(\mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha : i \to j} x_i x_j + \sum_{i,j \in Q_0} r_{ij} x_i x_j$$

Theorem 4.2:

The algebra $kQ/\langle R \rangle$ is

- of finite type if $q(\mathbf{x}) \geq 1$ for all $\mathbf{x}$;
- of tame type if $q(\mathbf{x}) \geq 0$ for all $\mathbf{x}$, and attains 0 somewhere;
- of wild type if $q(\mathbf{x}) < 0$ for some $\mathbf{x}$.

References

[CB88] W. W. Crawley-Boevey, *On tame algebras and bocses*, Proc. London Math. Soc. (3) **56** (1988), no. 3, 451–483.

[Dro80] Ju. A. Drozd, *Tame and wild matrix problems*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., vol. 832, Springer, Berlin, 1980, pp. 242–258.