An upper bound for the representation dimension of group algebras with elementary abelian Sylow p-subgroups

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Abstract. In this article we present an upper bound on the representation dimension of the group algebra of a group with an elementary abelian Sylow p-subgroup. Specifically, if k is a field of characteristic p and G is a group with elementary abelian Sylow p-subgroup P, we prove that the representation dimension of kG is bounded above by the order of P. Key to proving this theorem is the separable equivalence between the two algebras and some nice properties of Mackey decomposition.

1 Introduction

The representation dimension of a finite dimensional algebra over a field was introduced by Auslander in [Aus71] with the hope that it would measure how far an algebra was from being of finite representation type. Auslander showed that an algebra is of finite representation type if and only if its representation dimension is at most 2. For the past 50 years, representation dimension has proved very difficult to calculate in general, with most results offering only bounds on the dimension. Two major results came from Iyama in 2003 and Rouquier in 2006. Iyama showed in [Iya03] that representation dimension is always finite. In [Rou06], Rouquier gave the first example of an algebra with representation dimension greater than 3 and in the same article provided a family of algebras that demonstrate representation dimension can be arbitrarily large.

In this article we follow the ideas of Iyama, which were later refined by Ringel in [Rin10] and utilised by Bergh and Erdmann in [BE11]. We use these ideas to establish an upper bound for the representation dimension of certain group algebras kG. Specifically, if k is a field of characteristic p and G is a finite group with elementary abelian Sylow p-subgroup P, then we show that $\operatorname{rep\,dim} kG \leq |P|$.

[Aus71] Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes (1971), 70, Republished in [Aus99]

[Iya03] Iyama, *Finiteness of representation dimension*, Proc. Amer. Math. Soc. **131** (2003), no. 4, 1011–1014 (electronic)

[Rou06] Rouquier, *Representation dimension of exterior algebras*, Invent. Math. **165** (2006), no. 2, 357–367

[Rin10] Ringel, *Iyama's finiteness theorem via strongly quasi-hereditary algebras*, J. Pure Appl. Algebra **214** (2010), no. 9, 1687–1692

[BE11] Bergh and Erdmann, *The representation dimension of Hecke algebras and symmetric groups*, Adv. Math. **228** (2011), no. 4, 2503–2521

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Theorem 4:

> Let k be a field of characteristic p. If G is a finite group with elementary abelian Sylow-p subgroup P then
> $$\operatorname{rep\,dim} kG \leq |P|.$$

This article is structured as follows: in section 2 we provide the necessary preliminary definitions and results. In section 3 we present the ideas behind the proof and in section 4 we give the full details of the result.

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2 Preliminaries

2.1 Representation dimension

Definition (Projective dimension)**.** Let A be a finite dimensional algebra over a field, let M be a finitely generated (right) A-module and let

$$\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a projective resolution of M. We say that the resolution has *length* n if $P_n \neq 0$ but $P_i = 0$ for all $i > n$. If this property does not hold for any n, then the resolution is of *infinite length*.

The *projective dimension* of M, denoted by $\operatorname{pd}(M)$, is defined to be the minimal length of a projective resolution of M.

Definition (Global dimension)**.** Let A be a finite dimensional algebra over a field. The *global dimension* of A, denoted by $\operatorname{gl\,dim}(A)$, is defined to be the supremum of the projective dimensions of all finitely generated (right) A-modules.

$$\operatorname{gl\,dim}(A) = \sup \{\operatorname{pd}(M) \mid M \text{ an } A\text{-module}\}$$

Definition (Generator/cogenerator)**.** Let A be a finite dimensional algebra over a field. A module M is said to *generate* $\operatorname{mod} A$, the category of finitely generated right A-modules, if for any module $N \in \operatorname{mod} A$ there is a positive integer n and an epimorphism

$$M^n \longrightarrow N \longrightarrow 0.$$

A module M is said to *cogenerate* $\operatorname{mod} A$ if for any module $N \in \operatorname{mod} A$ there is a positive integer n and a monomorphism

$$0 \longrightarrow N \longrightarrow M^n.$$

Note that M is a generator of $\operatorname{mod} A$ if and only if M contains each finitely generated indecomposable projective A-module as a direct summand. Similarly, M is a cogenerator of $\operatorname{mod} A$ if and only if it contains each finitely generated indecomposable injective A-module as a direct summand. In the case of self-injective algebras, such as group algebras, these two properties are equivalent. For more details regarding generators and cogenerators, and their relationships to projective and injective modules see, for example, [AF92, sections 8, 17 and 18].

Definition (Representation dimension)**.** Let A be a finite dimensional algebra over a field. The representation dimension of A is defined by:

$$\operatorname{rep\,dim}(A) = \inf\left\{\operatorname{gl\,dim}\left(\operatorname{End}_A(M)\right)\,\middle|\, M \text{ generates and cogenerates } \operatorname{mod} A\right\}$$

If A is semisimple then each module is projective and hence $\operatorname{rep\,dim} A = 0$. Otherwise, Auslander showed in [Aus71] that $\operatorname{rep\,dim} A = 2$ if and only if A is of finite representation type; that is A has only a finite set of isomorphism classes of indecomposable modules.

2.2 Separable equivalence

Separable equivalence of finite dimensional algebras was introduced by Linckelmann in [Lin11] and Bergh and Erdmann first discussed separable division in [BE11] as a refinement of the idea. The same concept for rings was studied by Kadison in [Kad95] and [Kad17].

Definition (Separable division/equivalence)**.** Let A and B be finite dimensional algebras over a field. We say that A *separably divides* B if there are bimodules ${}_AM_B$ and ${}_BN_A$ such that

(a) the modules ${}_AM$, M_B, ${}_BN$ and N_A are finitely generated and projective; and

(b) there is a split bimodule epimorphism ${}_AM \underset{B}{\otimes} N_A \twoheadrightarrow {}_AA_A$

We say that A and B are *separably equivalent* if A separably divides B and B separably divides A.

If k is a field of characteristic p and G is a finite group with Sylow p-subgroup P, then kG is separably equivalent to kP. This fact was noted by Linckelmann in [Lin11] and the bimodules giving the equivalence are ${}_{kP}kG_{kG}$ and ${}_{kG}kG_{kP}$ (and the associated tensor functors give induction and restriction). This separable equivalence will play a major role in establishing an upper bound for the representation dimension of group algebras.

It was shown in [Pea17] that the representation type of an algebra is preserved under separable equivalence, but much less is known with regard to representation

[AF92] Anderson and Fuller, *Rings and categories of modules*, second ed., Graduate Texts in Mathematics, vol. 13, Springer-Verlag, New York, 1992

[Aus71] Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes (1971), 70, Republished in [Aus99]

[Lin11] Linckelmann, *Finite generation of Hochschild cohomology of Hecke algebras of finite classical type in characteristic zero*, Bull. Lond. Math. Soc. **43** (2011), no. 5, 871–885

[BE11] Bergh and Erdmann, *The representation dimension of Hecke algebras and symmetric groups*, Adv. Math. **228** (2011), no. 4, 2503–2521

[Kad95] Kadison, *On split, separable subalgebras with counitality condition*, Hokkaido Math. J. **24** (1995), no. 3, 527–549

[Kad17] Kadison, *Symmetric, separable equivalence of rings*, arχiv preprint arXiv:1710.09251 (2017), 1–9

[Pea17] Peacock, *Separable equivalence, complexity and representation type*, J. Algebra **490** (2017), 219–240

dimension. Bergh and Erdmann ([BE11]) used the separable equivalence between certain symmetric-group algebras and the group algebras for their Sylow p-subgroups to prove bounds on representation dimension must hold. Following this work (at a talk in 2015 [Ber15]) Bergh asked to what extent representation dimension is preserved under separable equivalence. Specifically Bergh asked the following two questions, both of which still remain open.

[BE11] Bergh and Erdmann, *The representation dimension of Hecke algebras and symmetric groups*, Adv. Math. **228** (2011), no. 4, 2503–2521

[Ber15] Bergh, *Representation dimension of finite-dimensional algebras*, 71st BLOC meeting (2015)

Question. If finite dimensional algebras A and B are separably equivalent does this mean that $\operatorname{rep\,dim} A = \operatorname{rep\,dim} B$?

Question. If finite dimensional algebras A and B are separably equivalent and $\operatorname{rep\,dim} A = 3$ does this mean that $\operatorname{rep\,dim} B = 3$?

Note that since separable equivalence is known to preserve representation type we also know that a representation dimension strictly less than three is also preserved.

3 Bounding representation dimension

Auslander showed in [Aus71] that the representation dimension of a selfinjective algebra is bounded above by the algebra's Loewy length, that is the length of its radical series. In this section we will establish a different upper bound for the representation dimension of a group algebra based only on the size of its Sylow subgroups. Specifically we will show that if k is a field of characteristic p and G is a finite group with elementary abelian Sylow p-subgroup P, then $\operatorname{rep\,dim} kG \leq |P|$. Note that if n is the rank of the elementary abelian p-group P, then the Loewy length of kP is $n(p-1)+1$, which is in general less than $|P| = p^n$. Thus in many cases Auslander's bound is better than the one we prove here; the advantage of our approach is that we no longer need to know anything directly about the group algebra kG, and so in particular may not know its Loewy length.

[Aus71] Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes (1971), 70, Republished in [Aus99]

We will establish the upper bound for representation dimension by providing an explicit construction of a generator M, and demonstrating that the global dimension of $\operatorname{End} M$ is less than or equal to the given bound. We will begin with some elementary definitions and an overview of the ideas behind the proof before giving the full details in section 4.

Definition (Additive closure)**.** Let A be a finite dimensional algebra over a field and let $\mathcal{M}$ be a set of finitely generated A-modules. We define the *additive closure* of $\mathcal{M}$, denoted $\operatorname{add}\mathcal{M}$, to be the full subcategory of $\operatorname{mod} A$ whose objects are finite direct sums of direct summands of modules in $\mathcal{M}$.

$$\operatorname{add}\mathcal{M} = \left\{ N \in \operatorname{mod} A \;\middle|\; \begin{array}{l} \text{there is a finite subset } \{M_1, \dots, M_r\} \subseteq \mathcal{M} \text{ and} \\ \text{positive integers } n_1, \dots, n_r \text{ such that } N \text{ is a direct} \\ \text{summand of } \bigoplus_{i=1}^{r} M_i^{n_i} \end{array} \right\}.$$

If $\mathcal{M} = \{M\}$ is a singleton set then we use the alternative notation $\operatorname{add} M = \operatorname{add}\mathcal{M}$.

Notation. Let $\mathcal{C}$ be a category. We denote the full category of indecomposable objects of $\mathcal{C}$ by $\operatorname{ind}\mathcal{C}$. If $\mathcal{M}$ is a set of finitely generated modules then we mirror the notation above and denote indecomposable objects of $\operatorname{add}\mathcal{M}$ by $\operatorname{ind}\mathcal{M}$. Similarly we denote the indecomposable summands of a module M by $\operatorname{ind} M$.

Theorem 1: *Bergh and Erdmann* [BE11, theorem 2.3]

> Let A and B be finite dimensional algebras over a field k and suppose there exists a B-module M such that
>
> (a) A separably divides B through ${}_A X_B$ and ${}_B Y_A$; and
>
> (b) $\operatorname{Hom}_A(Y, M \underset{B}{\otimes} Y) \in \operatorname{add}_B M$
>
> then $\operatorname{gl\,dim}\big(\operatorname{End}_A(M \underset{B}{\otimes} Y)\big) \le \operatorname{gl\,dim}(\operatorname{End}_B(M))$.

[BE11] Bergh and Erdmann, *The representation dimension of Hecke algebras and symmetric groups*, Adv. Math. **228** (2011), no. 4, 2503–2521

In our situation we have k a field of characteristic p, G a finite group and P a Sylow p-subgroup of G. Now in the language of Theorem 1 if we let

$$A = kG \qquad B = kP \qquad X = {}_{kG}kG_{kP} \qquad Y = {}_{kP}kG_{kG}$$

then Property (a) is immediate from Linckelmann's original observation in [Lin11]. In Property (b), $M \underset{kP}{\otimes} kG$ is simply induction $M\!\uparrow^{G}$ and $\operatorname{Hom}_{kG}\big({}_{kP}kG, N\big)$ is restriction $N\!\downarrow_{P}$ for any kP-module M and any kG-module N. We therefore have the following corollary.

[Lin11] Linckelmann, *Finite generation of Hochschild cohomology of Hecke algebras of finite classical type in characteristic zero*, Bull. Lond. Math. Soc. **43** (2011), no. 5, 871–885

Corollary. *Let P be a Sylow p-subgroup of G and k a field of characteristic p. If M is a kP-module such that $M\!\uparrow^{G}\!\downarrow_{P} \in \operatorname{add} M$ then*

$$\operatorname{gl\,dim}\big(\operatorname{End}_{kG}(M\!\uparrow^{G})\big) \le \operatorname{gl\,dim}(\operatorname{End}_{kP}(M)).$$

In light of this corollary, if we can find a generator M of kP, such that $\operatorname{add} M$ is closed under induction to any group G that contains P as a Sylow p-subgroup, and restriction back down to P, then the representation dimension of kG is bounded above by the global dimension of $\operatorname{End}_{kP}(M)$.

Definition (Restriction of scalars)**.** Recall that if $\phi\colon H \to L$ is a homomorphism of groups and M is an L-module then we may give M the structure of an H-module by *restriction of scalars* where we define the product $mh = m\phi(h)$ for all $m \in M$ and $h \in H$. We denote this L-module by $M\!\downarrow_{\phi}$. Note that if ϕ is the inclusion of a subgroup then $M\!\downarrow_{\phi} = M\!\downarrow_{H}$ is the usual definition of the restriction of a module to a subgroup.

Let us assume that for a p-group P we have a finite set of modules $\mathcal{M}_P$ with the properties:

(res-ind) if $X \in \mathcal{M}_P$ and L is a subgroup of P then $X\downarrow_L\uparrow^P \in \operatorname{add} \mathcal{M}_P$;

(isom) if H and L are subgroups of P and there is an isomorphism $\phi\colon H \xrightarrow{\sim} L$ then

$$\operatorname{ind}\big((\mathcal{M}_P\downarrow_L)\downarrow_\phi\big) = \operatorname{ind}\big(\mathcal{M}_P\downarrow_H\big),$$

where $\mathcal{M}_P\downarrow_H = \{X\downarrow_H \mid X \in \mathcal{M}_P\}$ and $(\mathcal{M}_P\downarrow_L)\downarrow_\phi = \{(X\downarrow_L)\downarrow_\phi \mid X \in \mathcal{M}_P\}$ are respectively the sets of modules obtained by restricting directly to H or by first restricting to L and then restricting to H via the isomorphism ϕ.

For any group G that contains P as a subgroup, Mackey decomposition gives us

$$M\uparrow^G\downarrow_P \cong \bigoplus_{s \in P\backslash G/P} (M \otimes s)\downarrow_{s^{-1}Ps\cap P}\uparrow^P$$

and so if

$$M = \bigoplus_{X \in \mathcal{M}_P} X$$

then the properties (res-ind) and (isom) mean that $\operatorname{add} M$ is closed under induction-restriction. If M is also a generator for kP then we can use the global dimension of $\operatorname{End}_{kP} M$ to simultaneously bound the representation dimension of all group algebras for groups with P as a Sylow p-subgroup.

4 Elementary abelian groups

In this section we will define a class of modules for elementary abelian p-groups that is closed under induction-restriction and that contains a generator of the group algebra (the regular module). Throughout this section we fix a prime p and a field k of characteristic p. Using the remarks made at the end of section 3 we will use this class of modules to bound the representation dimension of group algebras for all groups with the given elementary abelian p-group as a Sylow p-subgroup.

We begin with some notation and then describe the class of modules $\mathcal{M}_P$, which we alluded to in section 3.

Notation. Let N be a module with Loewy length $n \in \mathbb{N}$, denoted $\operatorname{Lowlen} N = n$. That is $\operatorname{rad}^n N = 0$ but $\operatorname{rad}^{n-1} N \neq 0$. For any positive integer n we denote by $N_{(m)}$ the quotient module

$$N_{(m)} = \frac{N}{\operatorname{rad}^m N}.$$

By convention we let $\operatorname{rad}^m N = N$ and $N_{(m)} = 0$ whenever $m \leq 0$. Notice that if $0 \leq m \leq n = \operatorname{Lowlen} N$ then the Loewy length of $N_{(m)}$ is m.

We denote by $\mathfrak{r}N$ the quotient module $\mathfrak{r}N = N_{(n-1)}$. That is, $\mathfrak{r}$ is an operator that reduces the Loewy length of a non-zero module by one.

Definition ($\mathcal{M}_P$)**.** Let P be an elementary abelian p-group and k a field of characteristic p. Let $\mathcal{M}_P$ be the set of indecomposable kP-modules that is minimal with respect to the following properties:

(a) $kP \in \mathcal{M}_P$;

(b) if $X \in \mathcal{M}_P$ and H is a subgroup of P then $\operatorname{ind}\big(X{\downarrow_H}{\uparrow^P}\big) \subseteq \mathcal{M}_P$;

(c) if $X \in \mathcal{M}_P$ and m is a positive integer then $\operatorname{ind}\big(X_{(m)}\big) \subseteq \mathcal{M}_P$.

Property (a) of this definition means that $\mathcal{M}_P$ contains a generator of kP and property (b) is simply stating that the class is closed under the (res-ind) property. To see that $\mathcal{M}_P$ is also closed under (isom) we first note that $\mathcal{M}_P$ is closed under automorphisms of P. Now an elementary abelian p-group is isomorphic to a vector space over the field of p elements and a subgroup is simply a subspace. To see that (isom) holds we use the fact that any isomorphism between subspaces of a vector space can be extended to an automorphism of the whole space; thus the same holds for P. If $\mathcal{M}_P$ is a finite set then we are in the position described at the end of section 3 and can use $\mathcal{M}_P$ to find an upper bound for the representation dimension of kG for any finite group G with a Sylow p-subgroup isomorphic to P.

We have not yet mentioned property (c) of the definition: this property means we obtain a strongly quasi-hereditary endomorphism ring. By a result of Ringel in [Rin10], this is known to have finite global dimension. If we excluded property (c) we could still calculate an upper bound for representation dimension, however in general this value would be infinite.

[Rin10] Ringel, *Iyama's finiteness theorem via strongly quasi-hereditary algebras*, J. Pure Appl. Algebra **214** (2010), no. 9, 1687–1692

4.1 Finiteness of $\mathcal{M}_P$

We continue the assumption that P is an elementary abelian p-group and first aim to show that $\mathcal{M}_P$ is a finite set. We will do this by defining a finite collection of modules $\mathcal{N}_P$ and by demonstrating that this is an alternative description of $\mathcal{M}_P$. We define $\mathcal{N}_P$ inductively: if P is the trivial group then $\mathcal{N}_P = \{k\}$, otherwise we define $\mathcal{N}_P$ by

$$\mathcal{N}_P = \left\{ \mathfrak{r}^i\big(X{\uparrow^P}\big) \;\middle|\; X \in \mathcal{N}_H \text{ with } |P : H| = p \text{ and } 0 \leq i < p \right\}$$

In order to show that $\mathcal{M}_P$ and $\mathcal{N}_P$ are the same set we require the next three lemmas.

Notation (Subgroup). We use the notation $H \leq G$ to mean H is a subgroup of G. In order to indicate that H is a proper subgroup of G we use the notation $H < G$.

Lemma 4.1. *Let H be a proper subgroup of the elementary abelian p-group P.*

If $X \in \mathcal{M}_H$ then $X{\uparrow^P} \in \mathcal{M}_P$.

Proof. As we are working in a p-group and each $X \in \mathcal{M}_H$ is indecomposable we have that $X{\uparrow}^P$ is also indecomposable. This is a simple application of Green's indecomposability theorem: see [Gre59] or [Ben91, theorem 3.13.3].

[Gre59] Green, *On the indecomposable representations of a finite group*, Math. Z. **70** (1959), 430–445

Next we note that each X in $\mathcal{M}_H$ can be obtained from $X_0 = kH$ after applying a finite number of steps $X_i \mapsto X_{i+1}$ where

[Ben91] Benson, *Representations and cohomology. I*, Cambridge Studies in Advanced Mathematics, vol. 30, Cambridge University Press, Cambridge, 1991

(a) X_{i+1} is a summand of $X_i{\downarrow}_L{\uparrow}^H$ for some subgroup $L < H$; or

(b) X_{i+1} is a summand of $(X_i)_{(m)}$ for some positive integer m.

It is clear that $kH{\uparrow}^P \in \mathcal{M}_P$ and we will prove the result by induction on the number of steps required to obtain X. We assume that X is obtain from Y in one step and that $Y{\uparrow}^P \in \mathcal{M}_P$.

(a) Let us assume that $L < H$ and X is an indecomposable summand of $Y{\downarrow}_L{\uparrow}^H$. We know by Mackey decomposition that Y is a summand of $Y{\uparrow}^P{\downarrow}_H$ and so $X{\uparrow}^P$ is a summand of $Y{\uparrow}^P{\downarrow}_L{\uparrow}^P$. All such summands are in $\mathcal{M}_P$ by the assumption on Y and the definition of $\mathcal{M}_P$.

(b) Assume that X is a summand of $Y_{(m)}$ for some positive integer m and note that without loss of generality we may assume that H is an index p subgroup of P:

$$H = \langle g_2, \ldots, g_n \rangle < \langle g_1, g_2, \ldots, g_n \rangle = P.$$

If we let $x = (g_1 - 1)$ then the induction of Y to P can be decomposed as

$$Y{\uparrow}^P \cong \bigoplus_{s=0}^{p-1} Y \underset{kH}{\otimes} x^s.$$

Similarly we have

$$\operatorname{rad}^m \left(Y{\uparrow}^P\right) \cong \bigoplus_{s=0}^{p-1} \operatorname{rad}^{m-s} Y \underset{kH}{\otimes} x^s.$$

We can put these together and get that

$$\left(Y{\uparrow}^P\right)_{(m)} \cong \bigoplus_{s=0}^{p-1} Y_{(m-s)} \underset{kH}{\otimes} x^s.$$

In particular $\left(\left(Y{\uparrow}^P\right)_{(m)}\right){\downarrow}_H$ contains $Y_{(m)}$ as a summand and therefore also X as a summand. That $X{\uparrow}^P \in \mathcal{M}_P$ is now immediate from the initial assumptions. □

Lemma 4.2. *Let H be a proper subgroup of the elementary abelian p-group P.*
If $X \in \mathcal{M}_P$ then $\operatorname{ind}(X\downarrow_H) \subseteq \mathcal{M}_H$.

Proof. We will follow a similar idea to the proof of Lemma 4.1.

The result is clear when $X = kP$ and so we assume that X is obtained in one step from $Y \in \mathcal{M}_P$ and that $\operatorname{ind}(Y\downarrow_L) \subseteq \mathcal{M}_L$ for any proper subgroup $L < P$.

(a) Let us assume that $L < P$ and X is an indecomposable summand of $Y\downarrow_L\uparrow^P$. Thus there is a module $Z \in \mathcal{M}_L$ such that $X \cong Z\uparrow^P$. By Mackey decomposition we then have

$$X\downarrow_H \;\cong\; Z\uparrow^P\downarrow_H \;\cong\; \underbrace{Z\downarrow_{L\cap H}\uparrow^H \oplus \cdots \oplus Z\downarrow_{L\cap H}\uparrow^H}_{|P:LH|\text{-copies}}$$

and summands of this are in $\mathcal{M}_H$ by the induction hypothesis and Lemma 4.1.

(b) Consider $X \cong Y_{(m)}$ for some positive integer m. Without loss of generality we may assume that there is a subgroup $L < P$ and $Y \cong Z\uparrow^P$ for some $Z \in \mathcal{M}_L$ and that both H and L are index p subgroups of P. We have

$$H = \langle h, g_3, \ldots, g_n \rangle$$
$$L = \langle l, g_3, \ldots, g_n \rangle$$

Suppose $H \neq L$ so that we may decompose $(kP)_{(m)}$ as L-H–bimodules

$$(kP)_{(m)} \cong \bigoplus_{i=0}^{p-1} (kL)_{(m-i)} \underset{k[L\cap H]}{\otimes} (h-1)^i.$$

Thus

$$\begin{aligned} X\downarrow_H &\cong Z \underset{kL}{\otimes} (kP)_{(m)}\downarrow_H \\ &\cong Z \underset{kL}{\otimes} \left(\bigoplus_{i=0}^{p-1} (kL)_{(m-i)} \underset{k[L\cap H]}{\otimes} (h-1)^i \right) \\ &\cong \bigoplus_{i=0}^{p-1} Z_{(m-i)} \underset{k[L\cap H]}{\otimes} (h-1)^i \end{aligned}$$

and so the result holds by the induction hypothesis. In the case that $H = L$ a similar argument applies. □

Lemma 4.3. *Let P be an elementary abelian p-group and let $H < P$ be a subgroup of index p. If X is a kH-module then for any positive integer m*

$$\left(X\uparrow^P\right)_{(m)} \cong \left(X_{(m)}\uparrow^P\right)_{(m)}$$

Proof. For a kP-module Y we have that $Y_{(m)} \cong Y \underset{kP}{\otimes} (kP)_{(m)}$ and so

$$\left(X_{(m)}\overset{P}{\uparrow}\right)_{(m)} \cong X \underset{kH}{\otimes} (kH)_{(m)} \underset{kH}{\otimes} (kP)_{(m)}$$

Fix $g \in P$ and consider the automorphism of H given by conjugation: $h \mapsto g^{-1}hg$. This map preserves $\mathrm{rad}(kH)$ for each $g \in P$ and therefore $\mathrm{rad}(kH)kP = kP\,\mathrm{rad}(kH)$ is a two-sided ideal of kP. This ideal is nilpotent (as $\mathrm{rad}(kH)$ is nilpotent) and hence must be contained in $\mathrm{rad}(kP)$. Similarly $\mathrm{rad}^m(kH) \subseteq \mathrm{rad}^m(kP)$ and so the map

$$\begin{array}{ccccc} (kH)_{(m)} & \underset{kH}{\otimes} & (kP)_{(m)} & \longrightarrow & (kP)_{(m)} \\ [h] & \otimes & [g] & \mapsto & [hg] \end{array}$$

is well-defined with inverse $[g] \mapsto 1 \otimes [g]$. We therefore have that

$$\left(X_{(m)}\overset{P}{\uparrow}\right)_{(m)} \cong X \underset{kH}{\otimes} (kH)_{(m)} \underset{kH}{\otimes} (kP)_{(m)} \cong X \underset{kH}{\otimes} (kP)_{(m)} \cong \left(X\overset{P}{\uparrow}\right)_{(m)}$$

□

Proposition. *Let P be an elementary abelian p-group.*

Then $\mathcal{N}_P = \mathcal{M}_P$.

Proof. It is clear that $\mathcal{N}_P = \mathcal{M}_P$ when P is the trivial group. We will proceed by induction on the rank of P.

From Lemma 4.1 we see that $\mathcal{N}_P \subseteq \mathcal{M}_P$, so we need only show that $\mathcal{N}_P$ is closed under the three properties defining $\mathcal{M}_P$.

Let $H < P$ be an index-p subgroup. Since $kH \in \mathcal{M}_H$ we have that $kP \in \mathcal{N}_P$. Next we consider the restriction-induction property. Given $X \in \mathcal{N}_P \subseteq \mathcal{M}_P$ we know by Lemma 4.2 that summands of $X\underset{L}{\downarrow}$ are in $\mathcal{M}_L$, we also have that there is an index p subgroup H of P with $L \leq H < P$ and by Lemma 4.1 summands of $X\underset{L}{\downarrow}\overset{H}{\uparrow}$ are in $\mathcal{M}_H$, thus we have that summands of $X\underset{L}{\downarrow}\overset{P}{\uparrow}$ are in $\mathcal{N}_P$.

Now we need only show that $\mathcal{N}_P$ is closed under taking quotients by powers of the radical. If $m = \mathrm{Lowlen}\, X$ is the Loewy length of X then $\mathrm{Lowlen}\, X\overset{P}{\uparrow} = m + p - 1$ and thus Lemma 4.3 tells us that

$$\mathfrak{r}^p\left(X\overset{P}{\uparrow}\right) = \left(X\overset{P}{\uparrow}\right)_{(m-1)} \cong \left((\mathfrak{r}X)\overset{P}{\uparrow}\right)_{(m-1)} = \mathfrak{r}^{p-1}\left((\mathfrak{r}X)\overset{P}{\uparrow}\right) \in \mathcal{N}_P$$

and similarly

$$\mathfrak{r}^{p+i}\left(X\overset{P}{\uparrow}\right) = \mathfrak{r}^{p-1}\left((\mathfrak{r}^{i+1}X)\overset{P}{\uparrow}\right) \in \mathcal{N}_P.$$

□

4.2 Bounding the global dimension

We have established that if P is an elementary abelian p-group then the set $\mathcal{M}_P$ is finite, thus we can define the kP-module $M = \bigoplus_{X \in \mathcal{M}_P} X$. We wish to find an upper bound for the global dimension of $\operatorname{End}_{kP} M$. This bound will come as a result of the algebra being strongly quasi-hereditary using a result of Ringel, which was based on ideas of Iyama: see [Rin10] and [Iya03].

[Rin10] Ringel, *Iyama's finiteness theorem via strongly quasi-hereditary algebras*, J. Pure Appl. Algebra **214** (2010), no. 9, 1687–1692

[Iya03] Iyama, *Finiteness of representation dimension*, Proc. Amer. Math. Soc. **131** (2003), no. 4, 1011–1014 (electronic)

Definition (Strongly quasi-hereditary)**.** Let Γ be a finite dimensional algebra over a field, let $\{S_i\}_{i \in I}$ be the set of simple modules and P_i the projective cover of S_i. We say that Γ is *left strongly quasi-hereditary with n layers* if there is a function ℓ (called the *layer function*)

$$\ell \colon I \to \{1, \ldots, n\}$$

such that for each simple module S_i, there is an exact sequence

$$0 \to R_i \to P_i \to \Delta_i \to 0$$

satisfying:

(a) $R_i = \bigoplus_{j \in J} P_j$ with $\ell(j) > \ell(i)$ for each $j \in J$;

(b) if S_j is a composition factor of $\operatorname{rad} \Delta_i$ then $\ell(j) < \ell(i)$.

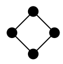
Theorem 2: [Rin10]

If Γ is a left strongly quasi-hereditary algebra with n layers then $\operatorname{gl dim}(\Gamma) \leq n$.

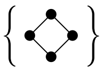
We will show that $(\operatorname{End}_{kP} M)^{\mathrm{op}}$ is left strongly quasi-hereditary by first defining a *layer function* on the elements of $\mathcal{M}_P$. This will directly transfer to a layer function on the projective (and therefore also the simple) modules of $(\operatorname{End}_{kP} M)^{\mathrm{op}}$.

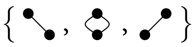
We define a partition of $\mathcal{M}_P$ inductively: first let $\mathcal{M}_P^0 = \{kP\}$. Now let $r_i = \max\{\operatorname{Lowlen} X \mid X \in \mathcal{M}_P \text{ but } X \notin \mathcal{M}_P^j \text{ for any } j < i\}$ be the maximum Loewy length of modules not yet included in a part. Let $d_i = \min\{\dim X \mid X \in \mathcal{M}_P \text{ and } \operatorname{Lowlen} X = r_i \text{ but } X \notin \mathcal{M}_P^j \text{ for any } j < i\}$ be the minimum dimension of modules of this Loewy length. Now we can define the next layer as $\mathcal{M}_P^i = \{X \in \mathcal{M}_P \mid \operatorname{Lowlen} X = r_i \text{ and } \dim X = d_i\}$.

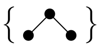
Example. We highlight this ordering with an example: let $P = C_2 \times C_2 = \langle g, h \rangle$. We have six modules in $\mathcal{M}_P$

Here we are denoting the regular module by  . Each edge  represents the action of $g - 1$ and each edge  represents the action of $h - 1$. The notation 

shows that g and h act in the same way. We use this notation as it nicely displays both the dimension and Loewy length of each module. The classes $\mathcal{M}_P^i$ are then given by

$\mathcal{M}_P^0 =$  Lowlen = 3, dim = 4

$\mathcal{M}_P^1 =$  Lowlen = 2, dim = 2

$\mathcal{M}_P^2 =$  Lowlen = 2, dim = 3

$\mathcal{M}_P^3 = \{\bullet\}$ Lowlen = 1, dim = 1

Theorem 3:

Let P be an elementary abelian p-group and let n be such that $\mathcal{M}_P^n$ is empty. If $M = \bigoplus_{X \in \mathcal{M}_P} X$ then $(\mathrm{End}_{kP} M)^{\mathrm{op}}$ is left strongly quasi-hereditary with at most n layers.

Proof. Let X be a module in $\mathcal{M}_P$ so that

$$P_X = \mathrm{Hom}_{kP}(X, M)$$

is an indecomposable projective $\mathrm{End}_{kP}(M)$-module and let

$$\pi\colon X \to \mathfrak{r}X$$

be the natural projection. Define Δ_X to be the quotient of $\mathrm{Hom}_{kP}(X, M)$ by those maps that factor through π:

$$\Delta_X = \frac{\mathrm{Hom}_{kP}(X, M)}{\{f \circ \pi \mid f\colon \mathfrak{r}X \to M\}}$$

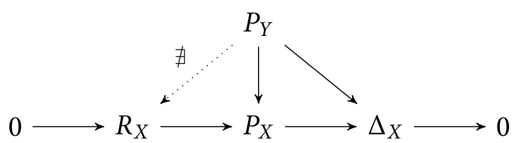
and let $R_X = \mathrm{Hom}(\mathfrak{r}X, M)$. We claim that the short exact sequence

$$0 \longrightarrow R_X \longrightarrow P_X \longrightarrow \Delta_X \longrightarrow 0$$

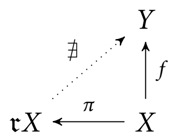
satisfies the properties in the definition of left strongly quasi-hereditary algebras.

(a) That R_X is projective and that if $X \in \mathcal{M}_P^i$ and $\mathfrak{r}X \in \mathcal{M}_P^j$ then $j > i$ is clear.

(b) Assume that the simple module corresponding to $Y \in \mathcal{M}_P^j$ is a composition factor of Δ_X. We have a map $P_Y \to \Delta_X$ that lifts to a map $P_Y \to P_X$ that does not factor through R_X:



By using the correspondence between $\operatorname{add} M$ and $(\operatorname{End}_{kP} M)^{\mathrm{op}}$ this gives a map $f\colon X \to Y$ that does not factor through π:



If $j > i$ then either $\operatorname{Lowlen} Y < \operatorname{Lowlen} X$, or the Loewy lengths are equal but $\dim Y > \dim X$. In either case if $m + 1 = \operatorname{Lowlen}(X)$ then $\operatorname{rad}^m X$ must be in the kernel of f.

Now assume that $j = i$ and f does not factor through π. In this situation the head of X maps onto the head of Y and since the dimensions of X and Y are equal, f must be an isomorphism.

This is enough to show that if Y is a composition factor of $\operatorname{rad} \Delta_X$ then $j < i$. □

Corollary. *Let P be an elementary abelian p-group and $M = \bigoplus_{X \in \mathcal{M}_P} X$. Then $\operatorname{gl\,dim}\big(\operatorname{End}_{kP}(M)\big) \leq |P|$.*

Proof. We need only establish that the number of distinct $(\operatorname{Lowlen}, \dim)$ pairs in $\mathcal{M}_P$ is bounded-above by p^r and this is certainly true when P is the trivial group. Now each module in $\mathcal{M}_P$ is one of p quotients of a module induced from an index-p subgroup. Thus the set of distinct pairs can only increase by a factor of at most p for each increase in rank. □

Theorem 4:

Let k be a field of characteristic p. If G is a finite group with elementary abelian Sylow-p subgroup P then

$$\operatorname{rep\,dim} kG \leq |P|.$$

Proof. Let M be as in the corollary to Theorem 3, then it follows that

$$\begin{aligned}
\operatorname{rep\,dim}(kG) &\leq \operatorname{gl\,dim}\big(\operatorname{End}_{kG}\big(M\!\uparrow^{G}\big)\big) && \text{by definition of representation dimension,}\\
&\leq \operatorname{gl\,dim}(\operatorname{End}_{kP} M) && \text{by the corollary to Theorem 1,}\\
&\leq |P| && \text{by the corollary to Theorem 3.}
\end{aligned}$$

□

5 Final remarks and calculations

We end with some comments on the tightness of the bound given by the theorem of the last section and some implications for other (non elementary-abelian) groups. Since the class of modules $\mathcal{M}_P$ has an explicit construction, for small enough groups we can simply construct the required module (say in *Magma*) and calculate the global dimension of its endomorphism ring. In fact, even if the group P is not elementary abelian we can still attempt to construct the class $\mathcal{M}_P$, however we no longer have a guarantee that it is finite. The first example of an abelian group for which $\mathcal{M}_P$ is infinite is the group $P = C_8 \times C_2 \times C_2$, but there are examples of non-abelian groups of order 16 with the same property. Even in the case that $\mathcal{M}_P$ is finite, without the implicit homogeneity we have with elementary abelian groups, the class need not satisfy the (isom) property; for example if $P = C_4 \times C_8$ then the (isom) property does not hold. In this case even if we can calculate the global dimension of the endomorphism ring, this value will not (necessarily) provide a bound on the representation dimension for all groups with Sylow p-subgroup isomorphic to P.

With the preceding remarks in mind, we have calculated the global dimension of the presented endomorphism ring for each abelian p-group of order 16 or less. This gives bounds for the representation dimension of any group algebra of a group with the given Sylow p-subgroup. For cyclic groups the global dimension is always 2, the remaining values can be found in the table below. As we can see from the values in this table, the bound given by Theorem 4 is not tight except in the case that $P = C_2$.

P	$C_2 \times C_2$	$C_4 \times C_2$	$C_2 \times C_2 \times C_2$	$C_3 \times C_3$	$C_8 \times C_2$
rep dim $\leq$	3	5	6	5	5

P	$C_4 \times C_4$	$C_4 \times C_2 \times C_2$	$C_2 \times C_2 \times C_2 \times C_2$
rep dim $\leq$	6	8	8

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