MATH20222: Introduction to Geometry

Sheet 1 Solutions — Semester 2 2020-21

Throughout $\mathbb{E}^n$ denotes an n-dimensional Euclidean vector space with orthonormal basis $\mathcal{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$.

1. Consider the following vectors in $\mathbb{R}^2$:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$

   (a) Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for $\mathbb{R}^2$.

   (b) Show that $\{\mathbf{a}, \mathbf{b}\}$ is a basis for $\mathbb{R}^2$.

   (c) Show that $\{\mathbf{e}_1, \mathbf{b}\}$ is *not* a basis for $\mathbb{R}^2$.

Solution: For each pair of vectors, we have to show they are linearly independent and span $\mathbb{R}^2$.

(a)

$$\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 = \mathbf{0} \ \Rightarrow \ \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ \Rightarrow \ \lambda_1 = \lambda_2 = 0$$

Therefore they are linearly independent. For any arbitrary vector $\mathbf{x} = (x_1, x_2)^\mathsf{T}$ of $\mathbb{R}^2$, we can write it as

$$\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

therefore $\{\mathbf{e}_1, \mathbf{e}_2\}$ span $\mathbb{R}^2$.

(b)

$$\lambda_1 \mathbf{a} + \lambda_2 \mathbf{b} = \mathbf{0} \ \Rightarrow \ \lambda_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\lambda_1 + 3\lambda_2 \\ 3\lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The second coordinate gives $\lambda_1 = 0$ and therefore $\lambda_2 = 0$, and so they are linearly independent. For any arbitrary vector $\mathbf{x} = (x_1, x_2)^\mathsf{T}$ of $\mathbb{R}^2$, we can write it as the linear combination

$$\lambda_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\Rightarrow \lambda_1 = \frac{x_2}{3}, \ \lambda_2 = \frac{x_1 - 2\lambda_1}{3} = \frac{x_1}{3} - \frac{2x_2}{9},$$

therefore $\{\mathbf{a}, \mathbf{b}\}$ span $\mathbb{R}^2$.

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(c)

$$\lambda_1\mathbf{e}_1 + \lambda_2\mathbf{b} = \mathbf{0} \;\Rightarrow\; \lambda_1 \begin{bmatrix}1\\0\end{bmatrix} + \lambda_2 \begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}\lambda_1 + 3\lambda_2\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

This holds for $\lambda_1 = 3, \lambda_2 = -1$, therefore we have a linear dependence with non-zero coefficients and so they do not form a basis.

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2. State whether each of the following maps $\langle -, - \rangle$ define an inner product on $\mathbb{R}^3$.

   [*where* $\mathbf{x} = (x_1, x_2, x_3)^\mathsf{T}, \mathbf{y} = (y_1, y_2, y_3)^\mathsf{T}$.]

   (a) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2$

   (b) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 3x_2 y_2 + 5x_3 y_3$

   (c) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_2 + x_2 y_1 + x_3 y_3$

   Solution:

   (a) $\langle -, - \rangle$ is not an inner product as it is not positive definite. To see this, $\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + x_2^2$, which is zero for certain non-zero vectors, such as $(0, 0, -1)^\mathsf{T}$.

   (b) $\langle -, - \rangle$ is an inner product. We need to check symmetry, linearity and that it is positive definite.

   $$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + 3x_2 y_2 + 5x_3 y_3 = y_1 x_1 + 3y_2 x_2 + 5y_3 x_3 = \langle \mathbf{y}, \mathbf{x} \rangle ,$$

   and so symmetry holds.

   $$\begin{aligned} \langle \lambda \mathbf{x} + \mu \mathbf{y}, \mathbf{z} \rangle &= (\lambda x_1 + \mu y_1) z_1 + 3(\lambda x_2 + \mu y_2) z_2 + 5(\lambda x_3 + \mu y_3) z_3 \\ &= \lambda (x_1 z_1 + 3x_2 z_2 + 5x_3 z_3) + \mu (y_1 z_1 + 3y_2 z_2 + 5y_3 z_3) \\ &= \lambda \langle \mathbf{x}, \mathbf{z} \rangle + \mu \langle \mathbf{y}, \mathbf{z} \rangle \end{aligned}$$

   and so linearity holds. Finally

   $$\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + 3x_2^2 + 5x_3^2 \geq 0$$

   is greater than or equal to zero for elements $\mathbf{x} \in \mathbb{R}^3$. Furthermore, there is equality if and only if $x_1^2 = 3x_2^2 = 5x_3^2 = 0$, which only occurs for $\mathbf{x} = \mathbf{0}$. Therefore positive-definite also holds.

   (c) $\langle -, - \rangle$ does not define an inner product as it does not satisfy positive definiteness:

   $$\langle \mathbf{x}, \mathbf{x} \rangle = 2x_1 x_2 + x_3^2$$

   takes negative values at $\mathbf{x} = (-1, 1, 0)^\mathsf{T}$.

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3. Let $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ be an (ordered) orthonormal basis of $\mathbb{E}^3$. Consider the ordered set of vectors $\mathcal{C} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ defined by $\mathcal{B}$ via:

   (a) $\mathbf{f}_1 = \mathbf{e}_2\,,\ \mathbf{f}_2 = \mathbf{e}_1\,,\ \mathbf{f}_3 = \mathbf{e}_3$

   (b) $\mathbf{f}_1 = \mathbf{e}_1\,,\ \mathbf{f}_2 = \mathbf{e}_1 + 3\mathbf{e}_3\,,\ \mathbf{f}_3 = \mathbf{e}_3$

   (c) $\mathbf{f}_1 = \mathbf{e}_1 - \mathbf{e}_2\,,\ \mathbf{f}_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2\,,\ \mathbf{f}_3 = \mathbf{e}_3$

   (d) $\mathbf{f}_1 = \mathbf{e}_2\,,\ \mathbf{f}_2 = \mathbf{e}_1\,,\ \mathbf{f}_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ where $\lambda \in \mathbb{R}$ is an arbitrary coefficient.

   For each set of vectors, write down the transition matrix from $\mathcal{B}$ to $\mathcal{C}$. Is $\mathcal{C}$ a basis? Is $\mathcal{C}$ orthogonal?

   Solution:

   (a)

$$ {}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \det({}_{\mathcal{B}}T_{\mathcal{C}}) = -1 $$

   The transition matrix has non-zero determinant, and so $\mathcal{C}$ is a basis. Moreover, we see $({}_{\mathcal{B}}T_{\mathcal{C}})^{\mathsf{T}}({}_{\mathcal{B}}T_{\mathcal{C}}) = I_3$ holds, and so the basis is orthogonal also.

   (b)

$$ {}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix} \quad , \quad \det({}_{\mathcal{B}}T_{\mathcal{C}}) = 0 $$

   As the transition matrix has determinant zero, $\mathcal{C}$ is not a basis. One could also note that the rank must be ≤ 2 as ${}_{\mathcal{B}}T_{\mathcal{C}}$ has a row of zeros. As the determinant is not equal to ± 1, it immediately cannot be orthogonal.

   (c)

$$ {}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \det({}_{\mathcal{B}}T_{\mathcal{C}}) = 0 $$

   As the transition matrix has determinant zero, $\mathcal{C}$ is not a basis. One could also notice the first two columns are proportional, and so there is a linear dependence between $\mathbf{f}_1, \mathbf{f}_2$. As the determinant is not equal to ± 1, it immediately cannot be orthogonal.

   (d)

$$ {}_{\mathcal{B}}T_{\mathcal{C}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad , \quad \det({}_{\mathcal{B}}T_{\mathcal{C}}) = -\lambda $$

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The transition matrix has nonzero determinant whenever $\lambda \neq 0$, and so $\mathcal{C}$ forms a basis when this occurs.

$\mathcal{C}$ is not orthogonal - if we calculate $({}_{\mathcal{B}}T_{\mathcal{C}})^{\mathsf{T}}({}_{\mathcal{B}}T_{\mathcal{C}})$, we see

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & \lambda \end{bmatrix} = \begin{bmatrix} 2 & 1 & \lambda \\ 1 & 2 & \lambda \\ \lambda & \lambda & \lambda^2 \end{bmatrix} \neq I_3.$$

One could also notice this by considering the length of $\mathbf{f}_3$:

$$|\mathbf{f}_3| = \sqrt{2+\lambda^2} > 1,$$

therefore $\mathcal{C}$ cannot be orthonormal as $\mathbf{f}_3$ is not unit length.

4. Consider the sets of polynomials

$$V = \left\{ ax^2 + bx + c \;\middle|\; a, b, c \in \mathbb{R} \right\} \;,\; T = \left\{ x^2 + px + q \;\middle|\; p, q \in \mathbb{R} \right\}$$

with the natural operations of addition and multiplication of polynomials.

[*You may assume these operations satisfy commutativity, associativity and distributivity.*]

(a) Which of these are vector spaces (over $\mathbb{R}$), and why?

(b) Show the polynomials $1, x, x^2$ are linearly independent in V.

(c) Calculate the dimension of V.

Solution:

(a) T is not a vector space, as the sum of any two polynomials does not belong to T e.g. $x^2 + x^2 = 2x^2 \notin T$.

V is a vector space. Let $f = a_2x^2 + a_1x + a_0, g = b_2x^2 + b_1x + b + 0$, to see the remaining axioms hold:

- (Zero) Setting $a_2 = a_1 = a_0 = 0$ gives $f = 0$ as the zero element.
- (Unity) Clearly $1 \cdot f = f$.
- (Additive inverses) For f, the inverse polynomial is $-f = (-a_2)x^2 + (-a_1)x + (-a_0)$.
- (Additive closure) The sum $f + g = (a_2 + b_2)x^2 + (a_1 + b_1)x + (a_0 + b_0)$ is contained in V.
- (Multiplicative closure) $\lambda \cdot f = (\lambda a_2)x^2 + (\lambda a_1)x + (\lambda a_0)$ is contained in V for any choice of $\lambda \in \mathbb{R}$.

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(b) Suppose $1, x, x^2$ satisfy an identity

$$c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot x^2 = \mathbf{0} (= 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2)$$

for some choice of $c_0, c_1, c_2 \in \mathbb{R}$. Equivalently, the polynomial $P(x) = c_0 + c_1 x + c_2 x^2$ equals zero for all choices of x. Testing this on the values $x = 0, 1, -1$, we see

$$\begin{aligned} P(0) &= c_0 \Rightarrow c_0 = 0 \\ P(1) &= c_1 + c_2 \Rightarrow c_1 = -c_2 \\ P(-1) &= -c_1 + c_2 \Rightarrow c_1 = c_2 \\ &\Rightarrow c_0 = c_1 = c_2 = 0 \end{aligned}$$

Therefore the only linear combination between $\{1, x, x^2\}$ equal to zero is when all coefficients are zero, and so they are linearly independent.

(c) $\{1, x, x^2\}$ are linearly independent and span V, therefore they form a basis. Therefore $\dim(V) = 3$.

5. Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be vectors of a vector space V. Show that if at least one of the vectors is equal to $\mathbf{0}$, then they are linearly dependent.

Solution: Without loss of generality, let us say $\mathbf{a}_1 = \mathbf{0}$. We can pick any arbitrary real $\lambda \neq 0$ such that

$$\lambda \cdot \mathbf{a}_1 + 0 \cdot \mathbf{a}_2 + \cdots + 0 \cdot \mathbf{a}_n = \mathbf{0}.$$

As we have found a linear combination where not all coefficients are zero, these vectors are linearly dependent.

6. Show that any three vectors in $\mathbb{R}^2$ must be linearly dependent.

Solution: Pick three arbitrary vectors

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Note that by Question 2, if any are equal to $\mathbf{0}$ then we are done. Therefore without

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loss of generality, we can say $a_1 \neq 0$. Consider the vectors

$$\mathbf{b}' = \mathbf{b} - \frac{b_1}{a_1}\mathbf{a} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \begin{bmatrix} b_1 \\ \frac{a_2 b_1}{a_1} \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_b \end{bmatrix}, \; \mu_b = b_2 - \frac{a_2 b_1}{a_1}$$

$$\mathbf{c}' = \mathbf{c} - \frac{c_1}{a_1}\mathbf{a} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} c_1 \\ \frac{a_2 c_1}{a_1} \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_c \end{bmatrix}, \; \mu_c = c_2 - \frac{a_2 c_1}{a_1}$$

If either $\mu_b, \mu_c = 0$, we have found a linear dependence between two of the vectors. Otherwise, $\mathbf{b}', \mathbf{c}'$ are linearly dependent as

$$\frac{1}{\mu_b}\mathbf{b}' - \frac{1}{\mu_c}\mathbf{c}' = \mathbf{0}.$$

This gives rise to a linear dependence between $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$:

$$\frac{1}{\mu_b}\left(\mathbf{b} - \frac{b_1}{a_1}\mathbf{a}\right) - \frac{1}{\mu_c}\left(\mathbf{c} - \frac{c_1}{a_1}\mathbf{a}\right) = \left(\frac{c_1}{a_1\mu_c} - \frac{b_1}{a_1\mu_b}\right)\mathbf{a} + \frac{1}{\mu_b}\mathbf{b} - \frac{1}{\mu_c}\mathbf{c} = \mathbf{0}.$$

7. Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be vectors of $\mathbb{E}^3$ such that each has unit length and they are pairwise orthogonal with each other. Show explicitly that they form a basis for $\mathbb{E}^3$.

   Solution: We use the proposition that n linearly independent vectors in $\mathbb{E}^n$ form a basis (proved in review session 1). As the space is three dimensional, it suffices to show the vectors are linearly independent, i.e. $\lambda_e\mathbf{e} + \lambda_f\mathbf{f} + \lambda_g\mathbf{g} = \mathbf{0}$ if and only if $\lambda_e = \lambda_f = \lambda_g = 0$. Take the inner product of $\lambda_e\mathbf{e} + \lambda_f\mathbf{f} + \lambda_g\mathbf{g}$ with $\mathbf{e}$:

$$\left\langle \lambda_e\mathbf{e} + \lambda_f\mathbf{f} + \lambda_g\mathbf{g}, \mathbf{e} \right\rangle = \lambda_e \langle \mathbf{e}, \mathbf{e} \rangle + \lambda_f \langle \mathbf{f}, \mathbf{e} \rangle + \lambda_g \langle \mathbf{g}, \mathbf{e} \rangle = \lambda_e \cdot 1 + \lambda_f \cdot 0 + \lambda_g \cdot 0 = 0$$

   implying $\lambda_e = 0$. Taking the inner product with $\mathbf{f}$ and $\mathbf{g}$ gives $\lambda_f = \lambda_g = 0$, therefore $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ are linearly independent.

8. Let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be vectors in $\mathbb{E}^3$ such that $\mathbf{a}, \mathbf{b}$ have unit length and are orthogonal to each other, and $\mathbf{c}$ has length $\sqrt{3}$ and forms the angle $\varphi = \arccos \frac{1}{\sqrt{3}}$ with $\mathbf{a}$ and $\mathbf{b}$. Show that $\{\mathbf{a}, \mathbf{b}, \mathbf{c} - \mathbf{a} - \mathbf{b}\}$ forms an orthonormal basis for $\mathbb{E}^3$.

   Solution: $\mathbf{a}, \mathbf{b}$ have unit length and are orthogonal to each other, we immediately have

$$\langle \mathbf{a}, \mathbf{a} \rangle = \langle \mathbf{b}, \mathbf{b} \rangle = 1, \; \langle \mathbf{a}, \mathbf{b} \rangle = 0.$$

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We also know that as $|\mathbf{c}| = \sqrt{\langle \mathbf{c}, \mathbf{c} \rangle} = \sqrt{3}$, we see that $\langle \mathbf{c}, \mathbf{c} \rangle = 3$. Furthermore, we can calculate $\langle \mathbf{a}, \mathbf{c} \rangle$ via

$$\langle \mathbf{a}, \mathbf{c} \rangle = |\mathbf{a}| \, |\mathbf{c}| \cos \varphi = 1 \cdot \sqrt{3} \cdot \frac{1}{\sqrt{3}} = 1.$$

We also get $\langle \mathbf{b}, \mathbf{c} \rangle = 1$ via the same calculation.

To show $\{\mathbf{a}, \mathbf{b}, \mathbf{c} - \mathbf{a} - \mathbf{b}\}$ form an orthonormal basis, we need to show they have unit length and are pairwise orthogonal. We know already $\mathbf{a}, \mathbf{b}$ are orthogonal and have unit length. To see $\mathbf{c} - \mathbf{a} - \mathbf{b}$ is orthogonal to $\mathbf{a}$ (and by a very similar calculation $\mathbf{b}$), we get

$$\langle \mathbf{c} - \mathbf{a} - \mathbf{b}, \mathbf{a} \rangle = \langle \mathbf{c}, \mathbf{a} \rangle - \langle \mathbf{a}, \mathbf{a} \rangle - \langle \mathbf{b}, \mathbf{a} \rangle = 1 - 0 - 1 = 0.$$

Finally, to check $\mathbf{c} - \mathbf{a} - \mathbf{b}$ has unit length, we get

$$\begin{aligned} \langle \mathbf{c} - \mathbf{a} - \mathbf{b}, \mathbf{c} - \mathbf{a} - \mathbf{b} \rangle &= \langle \mathbf{c}, \mathbf{c} \rangle + \langle \mathbf{a}, \mathbf{a} \rangle + \langle \mathbf{b}, \mathbf{b} \rangle + 2 \langle \mathbf{a}, \mathbf{b} \rangle - 2 \langle \mathbf{a}, \mathbf{c} \rangle - 2 \langle \mathbf{b}, \mathbf{c} \rangle \\ &= 3 + 1 + 1 + 2 \cdot 0 - 2 \cdot 1 - 2 \cdot 1 = 1, \end{aligned}$$

therefore $|\mathbf{c}| = \sqrt{\langle \mathbf{c}, \mathbf{c} \rangle} = 1$.

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